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Existence of Solutions of a Generalized Blasius Equation

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1. INTRODUCTION

In this paper we will show the existence of a unique solution of a boundary value problem for which the differential equation is

$$y''' + yy'' + f(y'^2) = 0 \tag{1}$$

The theorem established includes, as a special case, the existence and uniqueness of solutions of the Falkner–Skan problem, where $f(y'^2) = \lambda(1 - y'^2)$, $\lambda > 0$, and its special cases such as the Homan problem ($\lambda = \frac{1}{2}$). The Blasius problem, $\lambda = 0$, is omitted for convenience but, of course, it is well-known that it has a unique solution. So far as the special case of the Falkner–Skan problem, per se, is concerned the results of the paper are known. In this paper we establish a unique solution for a rather general boundary problem that includes problems frequently encountered in the study of boundary layer theory, the flow of fluids in general and, hopefully, elsewhere.

The book [13] of H. Schlichting is a classical reference to boundary layer theory. The mathematics of the Falkner–Skan problem prior to about 1963 is very well brought together by Philip Hartman [7]. Major papers prior to 1963 include the original Falkner–Skan paper [6], papers of H. Weyl [14], W. A. Coppel [5] (which was prepared independently of the significant papers [11, 12] of R. Iglisch) and a paper of Philip Hartman [8] devoted especially to the asymptotic behavior of solutions. Recent papers (of which [1, 2, 3, 4, 9, 10] are only a sample) have contributed to certain refinement questions such as reverse flow possibilities, further asymptotic properties, etc.

In the present paper we are concerned with the existence and uniqueness of solutions of (1) with boundary conditions

$$y(0) = \alpha, \qquad y'(0) = \beta, \qquad y'(\infty) = q \tag{2}$$

wherein $\alpha \geqslant 0$ and $p \leqslant \beta \leqslant q$. The numbers p and q are functions of $f(y'^2)$ of (1). In the Falkner–Skan case, $p = 0$, $q = 1$.

Clearly, the problem is not altered by the insertion of positive constants before

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any of the terms of (1) since the altered equation may be reduced to (1) by a change of variables.

The proof we give below is suggested by that of W. A. Coppel [5].

2. Existence-uniqueness Result

We prove the following theorem.

THEOREM. *Consider equation* (1) *wherein* $f(y'^2)$ *satisfies the conditions*

(a) $f(y'^2) > 0$ *for* $0 < p \leqslant y' < q$,

(b) $f(q^2) = 0$

(c) f *is continuous and monotone decreasing for* $0 < p \leqslant y' < q$,

(d) f *is sufficiently smooth to insure unique solutions of* (1) *and continuity in initial conditions. Let* $\alpha \geqslant 0$ *and* $p \leqslant \beta \leqslant q$ *be given numbers, then there exists a unique solution of* (1) *for which* (2) *is valid. For this solution* $y'' > 0$.

Proof. We write (1) as the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= -y_1 y_3 - f({y_2}^2) \end{aligned} \tag{3}$$

and define the sets

$$\begin{aligned} D &= \{(y_1, y_2, y_3) \mid y_1 > 0, p < y_2 < q, y_3 > 0\}, \\ D_a &= \{(y_1, y_2, y_3) \mid y_1 > 0, y_2 = p, y_3 > 0\}, \\ D_b &= \{(y_1, y_2, y_3) \mid y_1 > 0, p < y_2 < q, y_3 = 0\}. \end{aligned}$$

We now consider any triple of reals (α, β, γ) for which $p < \beta < q$, $\gamma \geqslant 0$. Note that

$$y_1 = qt + \alpha, \qquad y_2 = q, \qquad y_3 = 0 \tag{4}$$

is a solution of (3). We will depend upon the uniqueness of solutions to assure us that no solution of this autonomous system cuts the special solution (4).

Suppose that a solution corresponding to $y(0) = \alpha$, $y'(0) = \beta$ stays in D. In this event we now show that all conditions of (2) are satisfied (and clearly that y'', which in some models is the skin friction, is positive) and so the objective of the existence proof will be to show that this case does, indeed, occur. Under the assumptions of this paragraph, $\gamma > 0$ otherwise the solution woudl go out of D via D_b. We now observe that (i) $y_2' = y_3$ hence y_2 is increasing but y_2' is decreasing because $y_3' < 0$. Since the solution is assumed to stay in D, y_2 is bounded hence

$y_3 \to 0$. Since the solution is assumed to stay in D, y_2 is bounded hence $y_3 \to 0$ and, as a consequence, $y'' \to 0$ with this orientation. (ii) y_3 is clearly bounded between 0 and γ. (iii) $y_3' < -f(y_2^2)$ hence $y_2 \to q$ (i.e. $y'(\infty) = q$) otherwise the solution will go through D_b and, finally, (iv) $y_1' = y_2$ implies $y_1 \to \infty$.

Now, we wish to show that some solutions corresponding to (α, β, γ), γ variable, leave D via D_a and some leave via D_b. Since the solutions, as a function of γ, vary continuously and since no solution leaves via the edge (4) we may then conclude that some solution stays in D, the favorable case.

As for D_b, for the given α, β there exist values of γ for which the solutions leave via D_b since this is the case for $\gamma = 0$ and hence, by continuity, must be so for small positive γ.

Now, consider α, β (still fixed) and large values of γ. We will show for γ sufficiently large, solutions leave through D_a. One has

$$\begin{aligned} y_3' &= -(y_1 y_2)' + y_2^2 - f(y_2^2) \\ &\geqslant -(y_1 y_2)' + \beta^2 - f(\beta^2). \end{aligned}$$

Upon integration one secures

$$y_3 \geqslant \gamma + \alpha\beta - y_1(t)\, y_2(t) + \beta^2 t - f(\beta^2)\, t$$

In D, $p < y_2 < q$, hence $\alpha + pt \leqslant y_1 \leqslant qt + \alpha$ and

$$\begin{aligned} y_3 &\geqslant \gamma + \alpha\beta - q(qt + \alpha) + \beta^2 t - f(\beta^2)\, t \\ &= \gamma + \alpha\beta - q\alpha - (q^2 - \beta^2 + f(\beta^2))\, t \end{aligned}$$

It may be evident that if γ, which may vary, is sufficiently large then the solution does, in fact, leave D via D_a. Even so, we give this argument to be certain. We seek a $\bar{t}$ so that for all $0 \leqslant t \leqslant \bar{t}$ one has $y_3(t) > 0$ and $y_2(\bar{t}) \geqslant q$. Let $K = \gamma + \alpha\beta - q\alpha$, $S = q^2 - \beta^2 - f(\beta^2)$. Then $y_3 \geqslant K - St$, $y_2' = y_3 \geqslant K - St$ or $y_2 \geqslant Kt - St^2/2 + \beta$ (K being variable). To assure that

$$\begin{aligned} Kt - \frac{St^2}{2} + \beta &\geqslant q \\ K - St &> 0 \end{aligned}$$

simultaneously we notice that if $t > K/S$ the second inequality is satisfied. If $t = K/S + \epsilon$, the first inequality becomes

$$\frac{K^2}{S} - \frac{K^2}{2S} - \frac{\epsilon^2 S}{2} + \beta \geqslant q$$

from which it is clear that for any $\epsilon > 0$ and $K^2 \geqslant 2Sq + \epsilon^2 S - 2S\beta$ the desired conditions are satisfied.

This completes the existence part of the theorem and we have solutions corresponding to a connected set of γ's. We now show that the solution is unique.

Suppose that there are two solutions of (1) for which (2) is valid. We may assume that they are in D. That is, $y > 0$, $0 < p < y' < q$ and $y'' > 0$. In (1) let $y' = z$

$$y'' = z \frac{dz}{dy} \qquad \text{and} \qquad y''' = z \left(\frac{dz}{dy}\right)^2 + z^2 \frac{d^2z}{dy^2}$$

to secure

$$z'' = -\frac{1}{z}\left(\frac{dz}{dy}\right)^2 - \frac{y}{2}\left(\frac{dz}{dy}\right) - \frac{1}{z^2} f(z^2) \equiv F(y, z, z'). \tag{5}$$

If there are two solutions, then they give, for the equation (5), two solutions $z(y)$, $w(y)$ such that $z(\alpha) = \beta$, $z(\infty) = q$ and $w(\alpha) = \beta$, $w(\infty) = q$. Suppose $w(y) \neq z(y)$ for some $y > \alpha$. Since $z(\alpha) = w(\alpha)$ and $z(\infty) = w(\infty)$ we have the curves coinciding at $y = \alpha$ and having the same limit as $y \to \ y$. At some point one function, say $w(y)$, is above the other. Hence $v(y) = w(y) - z(y)$ has a positive maximum at, say, $y = c$. We have, $w(c) > z(c)$, $v(c) > 0$, $v'(c) = 0 = w'(c) - z'(c)$ hance $w'(c) = z'(c)$. At $y = c$, $v''(c) \leqslant 0$.

Now, $F(y, z, z')$ is increasing in its variable z since

$$y \frac{dz}{dy} = y \frac{dy'}{dy} = \frac{y}{z} y'' > 0.$$

Hence

$$v''(c) = F(\alpha, w(c), w'(c)) - F(\alpha, z(c), z'(c)) = w'(c) > 0$$

which is a contradiction and the proof is complete.

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